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Keplerian Representation of a Non-Keplerian Orbit

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The properties of a mean reference orbit are developed through the introduction of an augmented radial potential proportional to c/r^2. The arbitrary constant c is chosen to satisfy the perturbed energy level over an oblate Earth. This produces a mean orbit with not only the correct first order anomalistic period, but one which is also functionally identical to a Kepler orbit. Explicit formulas are obtained for the mean elements in terms of the osculating initial values. A comparison of the solution with a numerically integrated 12 h highly eccentric orbit shows that most of the error in the mean orbit is contained within the neglected periodic terms.

Introduction

THE fundamental problem in Keplerian representations of real orbits is the failure to account for the correct energy in the orbit. This leads to secular in-track errors (or equivalently timing errors) of the form, $\delta M = (n_k - n)t$, where n_k is the Kepler mean motion and n the real mean motion. As a consequence of δM, perturbations with respect to a reference orbit are not uniformly valid over long intervals of time.

A classical approach in perturbation theory is to incorporate as much as possible of the perturbative influence into a reference orbit that is still expressible in closed form. Garfinkel,[1] Sterne,[2] and Aksnes[3] employ an augmented reference potential that is a function of both radius and latitude. Vinti[4] utilizes oblate-spheroidal coordinates that lead to an exact solution for J_2 and part of J_4. Unfortunately, these solutions introduce elliptic functions and integrals and are somewhat more complicated than pure Kepler motion. If the augmented potential is restricted to a term proportional to c/r^2 (where c is a constant) then the solution retains the character of the Kepler motion. By suitable choice of this constant, we bring the orbit to nearly the correct energy level and remove the objectionable secular in-track drift.

The use of the energy integral for in-track stabilization is not new. Both Nacozy[5] and Baumgarte[6] have considered these ideas from the viewpoint of time transformations that incorporate the energy integral. Nacozy[7] outlines a method which constrains the numerical solution of a system of differential equations to remain on an integral surface by applying corrections at each integration step. This procedure is shown to reduce the errors in a two body numerical integration by several orders of magnitude. There are a series of papers on this topic by Baumgarte and Nacozy that generally treat the numerical aspects of stabilization of dynamical systems. The intent here is to present a simple closed form incorporation of the energy integral within the structure of Keplerian mechanics. This solution should prove to be of great utility in orbit transfer and targeting problems.

Orbit Equations

The equations of motion in the radial and transverse directions under the influence of a radial potential are given by

$$\frac{d^2u}{dv^2} + u = \frac{-f(1/u)}{h^2u^2} \tag{1}$$

$$\frac{d(r^2\dot{v})}{dt} = 0 \qquad r^2\frac{dv}{dt} = h \tag{2}$$

Equation (1) has been transformed in the usual way by the introduction of $u = 1/r$ and the change of the independent variable from time to true anomaly v. The angular momentum h is a constant of the motion for central forces. The function, $f(1/u)$, is the acceleration in the radial direction

$$f(r) = -\mu/r^2 + c/r^3 \tag{3}$$

where μ is the Earth's gravitational constant and we have augmented the spherical term by an arbitrary constant c proportional to r^{-3}.

Introducing

$$\alpha^2 = 1 + c/h^2 \tag{4}$$

and

$$\bar{u} = u - \mu/h^2\alpha^2 \tag{5}$$

into Eq. (1) brings it into the form,

$$\frac{d^2\bar{u}}{dv^2} + \alpha^2\bar{u} = 0 \tag{6}$$

The solution of Eq. (6) subject to the initial conditions, $\bar{u} = \bar{u}_0$ and $\bar{u}' = \bar{u}_0'$ at $v = v_0$, has the general form,

$$\bar{u} = \bar{u}_0 \cos\alpha(v - v_0) + (\bar{u}_0'/\alpha)\sin\alpha(v - v_0) \tag{7}$$

where

$$(h^2/\mu)\alpha^2\bar{u}_0 = \alpha^2(1 + e_0 \cos v_0) - 1 \tag{8}$$

$$(h^2/\mu)\alpha\bar{u}_0' = -(e_0\alpha \sin v_0) \tag{9}$$

The prime denotes differentiation with respect to true anomaly. e_0 and v_0 are the initial Kepler eccentricity and true anomaly, respectively. If we define ϵ and δ by

$$\epsilon \cos\delta = \alpha^2(1 + e_0 \cos v_0) - 1 \tag{10}$$

$$\epsilon \sin\delta = -e_0\alpha \sin v_0 \tag{11}$$

so that,

$$\epsilon = [(\epsilon\cos\delta)^2 + (\epsilon\sin\delta)^2]^{1/2} \tag{12}$$

and combine Eqs. (8-12) with Eq. (7), the orbit equation is brought into the standard form,

$$r = \frac{\alpha^2h^2/\mu}{1 + \epsilon\cos[\alpha(v - v_0) - \delta]} \tag{13}$$

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We deduce from Eq. (13) the eccentricity ϵ and semi-major axis a_m in the perturbed orbit,

$$\epsilon=\{[\alpha^2(1+e_0\cos v_0)-1]^2+(e_0\alpha\sin v_0)^2\}^{1/2} \tag{14}$$

$$a_m=\frac{\alpha^2h^2/\mu}{1-\epsilon^2}=a_0\left[1-\frac{(1+e_0\cos v_0)^2}{(1-e_0^2)}\left(\frac{c}{h^2}\right)\right]^{-1} \tag{15}$$

The quantity a_0 is the osculating semi-major axis at the epoch.

Time in the perturbed orbit follows from Eqs. (2) and (13)

$$t=\frac{p_0^2}{h}\alpha^4\int\frac{dv}{\{1+\epsilon\cos[\alpha(v-v_0)-\delta]\}^2}+C_1 \tag{16}$$

where p_0 is the Kepler semi-latus rectum and C_1 is the constant of integration. Let

$$\beta=\alpha(v-v_0)-\delta \tag{17}$$

and define an "eccentric anomaly" E_m by

$$\sin\beta=\frac{\sin E_m(1-\epsilon^2)^{1/2}}{1-\epsilon\cos E_m} \tag{18}$$

$$\cos\beta=\frac{-(\epsilon-\cos E_m)}{1-\epsilon\cos E_m} \tag{19}$$

Using Eqs. (17-19) in (16) gives

$$t=\frac{\alpha^3}{n_0}\left(\frac{1-e_0^2}{1-\epsilon^2}\right)^{3/2}\int(1-\epsilon\cos E_m)dE_m+C_1 \tag{20}$$

which integrates to the functional form of the familiar Kepler equation

$$t=\frac{1}{n_0}\left(\frac{a_m}{a_0}\right)^{3/2}(E_m-\epsilon\sin E_m)+C_1 \tag{21}$$

where n_0 is the Kepler mean motion, $n_0=(\mu/a_0^3)^{1/2}$. If the initial point is at perigee, then

$$\delta=0 \qquad \beta_0=0 \qquad C_1=0$$

In summary, we note that a perturbed orbit has been defined by an augmented radial potential that is specified by an arbitrary constant c. This orbit is completely defined by the osculating Kepler values a_0, e_0, v_0 at the epoch and the arbitrary constant c/h^2. It has the property of a slow rotation about the angular momentum vector with respect to the Kepler orbit. This follows by differentiating Eq. (17) with respect to time $[\dot\beta=\alpha\dot v\cong(1+c/2h^2)\dot v]$ to get

$$\dot\beta-\dot v\cong \tfrac{1}{2}(c/h^2)\dot v \tag{22}$$

It is also of interest to observe from Eq. (14) that a circular Kepler orbit in the presence of this potential has a nonzero eccentricity of order c/h^2.

Equation (15) shows that the constant c/h^2 is proportional to the energy in the perturbed orbit and provides the mechanism to choose its value. c/h^2 may be related to the anomalistic period by substituting a_m/a_0 from Eq. (15) into Eq. (21) with $E_m=2\pi$ to give

$$\frac{P_a}{P_0}=\left[1-\frac{(1+e_0\cos v_0)^2}{(1-e_0^2)}\left(\frac{c}{h^2}\right)\right]^{-3/2}$$

$$\cong 1+\frac{3}{2}\frac{(1+e_0\cos v_0)^2}{(1-e_0^2)}\left(\frac{c}{h^2}\right)+\text{order}, \theta\left(\frac{c}{h^2}\right)^2 \tag{23}$$

where P_a is the anomalistic period and P_0 is the Kepler period. This result can then be compared with any derivation of P_a/P_0, to obtain the coefficient, c/h^2. For example, we can use the anomalistic period over an oblate Earth from Claus and Lubowe[8] (or Sterne,[2] Eq. 5.2-7),

$$P_a/P_0=1-3/2\,J_2(a_e^2/r_0^3)a_0[1-3\sin^2\varphi_0]+\text{order}, \theta(J_2)^2 \tag{24}$$

Combining Eqs. (23) and (24) and solving for c/h^2,

$$\frac{c}{h^2}=-J_2\left(\frac{a_e}{r_0}\right)\left(\frac{p_0}{r_0}\right)\frac{(1-3\sin^2\varphi_0)}{(1+e_0\cos v_0)^2} \tag{25}$$

a_e is the Earth's equatorial radius and φ_0 is the initial geocentric latitude.

Finally we note that the energy integral can be obtained for any potential that does not contain time explicitly

$$V^2/2-U_{(r,\varphi)}=\bar C \tag{26}$$

where

$$U_{(r,\varphi)}=\frac{\mu}{r}\left\{1-\sum_{n=2}^{\infty}J_n\left(\frac{a_e}{r}\right)^nP_n(\sin\varphi)\right\} \tag{27}$$

and $P_n(\sin\varphi)$ is the Legendre polynomial, $\bar C$ the constant of integration, φ the latitude, and V the total velocity. Let

$$\bar C=-\mu/2a_m=V_0^2/2-U_{(r_0,\varphi_0)} \tag{28}$$

and since

$$V_0^2/2-\mu/r_0=-\mu/2a_0 \tag{29}$$

it follows from Eq. (28) that

$$\begin{aligned}1/a_m&=1/a_0+J_2(a_e^2/r_0^3)(1-\sin^2\varphi_0)\\&+J_3(a_e^3/r_0^4)\sin\varphi_0(3-5\sin^2\varphi_0)\\&\ldots+2J_2(a_e^n/r_0^{n+1})P_n(\sin\varphi_0)\end{aligned} \tag{30}$$

The constant c/h^2 may then be calculated from Eq. (15). The relationship of c/h^2 to the orbital energy is quite clear from this latter determination. The initial tangential velocity V_m in the mean orbit over a spherical Earth may be obtained from

$$V_m^2/2-\mu/r_0=-\mu/2a_m \tag{31}$$

where r_0 is the radius of the satellite at the epoch.

Some Numerical Results

As an example, let us compare a Kepler representation of the following typical 12 h high eccentricity orbit:

a_0	= 14272.41091 n. mi.
e_0	= 0.7340679052
i_0	= 62.9485 deg
Ω_0	= 292.286 deg
ω_0	= 271.349 deg
M_0	= 0 deg ($v_0=0$)
Epoch	= 1977 yr, 12 month, 13 days, 13 h, 14 min, 23.24577s (GMT)

Over a limited time span we will only consider the perturbations due to zonal harmonics, J_2-J_4, together with lunisolar gravitational effects.

For the mean orbit, use only the perturbation due to J_2 so that c/h^2 is given by Eq. (22). The above elements can be used

Table 1 Comparison of periods

	Anomalistic period, min
Numerical integration	717.733
Mean orbit	717.7673
Kepler orbit	712.7989

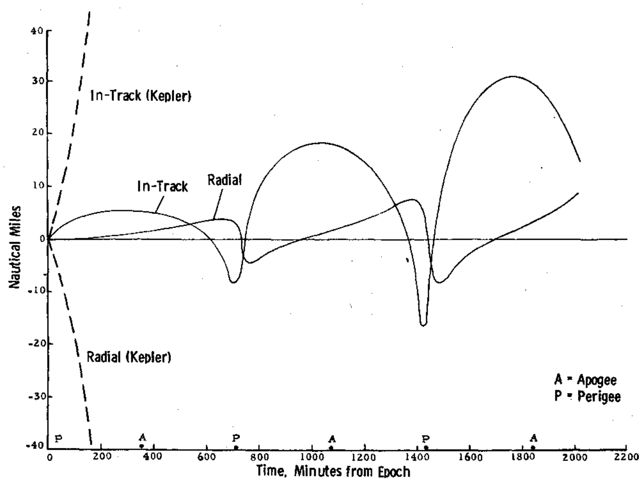

Fig. 1 In-plane errors of mean orbit.

directly in Eq. (22). The latitude of perigee is given by

$$\sin\varphi_0 = \sin i_0 \sin(\omega_0 + v_0) \qquad \varphi_0 = -62.91743265 \text{ deg}$$

The radius of perigee is

$$r_0 = a_0(1 - e_0) = 3795.4922 \text{ n. mi.}$$

With $a_e = 3443.9336$ n. mi. and $J_2 = 1082.76 \times 10^{-6}$, c/h^2 is calculated to be $+7.085072714 \times 10^{-4}$. Then from Eqs. (14) and (15),

$$\epsilon = 0.7352965090 \qquad a_m = 14338.655 \text{ n. mi.}$$

These values are then used in place of e_0, a_0 in the previous element set.

The mean a_m and ϵ together with the other elements are used as initial conditions to integrate the orbit over a spherical Earth potential. This reference orbit is then subtracted from a second orbit that is integrated under the perturbative influences of J_2-J_4 and the Sun and Moon using the osculating elements as initial conditions. For the Kepler compraison, we use the osculating initial conditions to integrate the orbit over a spherical Earth potential to obtain the reference orbit.

Table 1 compares the period obtained by numerical integration with the mean orbit and the Kepler orbit. The mean orbit has a period accurate to within about 2.0 s, which is quite good considering that only the effects of J_2 were used in deriving the mean orbit.

Figure 1 compares the radial and in-track errors from the mean orbit with the errors resulting from a Kepler orbit. It is seen that the Kepler orbit is grossly in error after only 200 min whereas the mean orbit has a maximum error of only 30 mi. after 2000 min. This error is growing in a secular manner, as is expected since the mean orbit does not have the exact real world period. The maximum radial and in-track errors over a period of 30 revs are 104.1 n. mi. and 347.0 n. mi., respectively.

This example is a severe test since the high eccentricity tends to magnify the amplitude of the short periodic effects. These effects are of course not modeled by the present analysis and constitute the major source of error. For low-altitude low-eccentricity orbits, the errors are typically less than several miles even in the presence of the dissipative mechanism due to drag. For example, using the foregoing procedure (J_2 only) for the low-altitude orbit, ($h_{p0} = 200$ n. mi., $e_0 = 0.001$, $i_0 = 96$ deg, $\omega_0 = \Omega_0 = M_0 = 0$) with a $W/C_DA = 50$ psf and a static ARDC 1959 atmosphere, the short periodic errors over a duration of 1440 min are bounded by

$$0.0989 \text{ n. mi.} < \delta r < 1.905 \text{ n. mi. (radial)}$$

$$-3.405 \text{ n. mi.} < r\delta\theta < 1.680 \text{ n. mi. (in-track)}$$

Concluding Remarks

The method outlined here gives a simple and direct way to determine a set of mean elements to match the energy of a perturbed orbit. Furthermore, these elements are used with a set of equations that are functionally identical to the familiar Kepler orbit equations. It has been shown that the use of this mean orbit dramatically reduces the errors with respect to a real orbit when compared with a Kepler representation of the same orbit.

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